On Time Dilation

An Essay by
Max Griffin

The basic idea of *time dilation* is usually described by saying something like "clocks slow down" when your velocity approaches the speed of light. That's kind of true, but it's also kind of not true. That facile explanation can lead to lots of confusion. More on that in a bit.

Let's start with the fact that time dilation is an undeniable feature of the universe, one that we use every day without realizing it. GPS systems have to correct for the time dilation caused by the relative motion of satellites and points on earth. Both the satellite sending signals and the receiver on earth are in motion. We'll see an example of this in a thought experiment at the end of this essay. The satellite and the earth-bound reciever are moving relative to each other due to the satellite's orbital velocity; that relative difference is what creates the time dilation. If the software didn't correct for that dilation, GPS systems would be off by many yards instead of a few feet.

The point is that time dilation is a part of our everyday life, even if we didn't know it until now.

What the Math Says.

The basic equation for time dilation, known as the *Lorentz Equation*, is as simple as the more famous $E = mc^2$. It's usually written something like this:

$$t_m = \gamma t_s$$

Of course, that's not useful unless you know what the letters stand for:

$$\begin{aligned} t_m &= \text{time as measured by the moving observer;} \\ t_s &= \text{time measured as by the stationary observer;} \\ \gamma &= \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \quad = \quad \text{the } \textit{Lorentz constant}\text{; where} \\ v &= \text{velocity of the moving observer; and} \\ c &= \text{the speed of light} \end{aligned}$$

The denominator of γ is always less than one, so γ is always greater than one. Thus, t_m, time as measured by the observer who is in motion, is always greater than t_s, time as measured by the stationary observer. Thus, the observer in motion sees a longer duration between events occuring in the stationary frame than does an observer in the stationary frame. Since time in this case is understood to be duration, this means that the observed duration is *longer* in the moving frame, i.e., that time has slowed down.

Most people think that derivation of the above formula involves deep mathematics. That's certainly true in general. However, we'll do a thought experiment at the end of this essay to deduce the formula in a special case using only Euclidian geometry.

Before we do that there is a problem with the above facile description. It's not that it's wrong, but rather that it's incomplete.

An Example.

The above summary has some missing elements, so that's where we'll turn next.

The first thing that's missing is *what*, exactly, are the two observers measuring? The above summary uses the word "events." That means that time, t, in the above equation is actually a measurement of *duration*. That generally means that, lurking in the background, are two events separated by time. That's in addition to the two observers. The observers are measuring the duration, from their perspective, of the *elapsed time* between the two events. That's the t_m and t_s in the equations, the *durations* that the moving observer and the stationary observer see.

Everything is moving, so the next thing that's missing is how we select the stationary and moving observer. Is it completely arbitrary which is which?

An example might be helpful. Let's suppose one of our observers is named, oh, I don't know, let's say, *Spock*, and he's on the Enterprise. The other observer might be *Sarin*, on a *different* spaceship. Both spaceships are moving really fast, either toward or away from each other. Do we get to choose which one is the stationary observer? Well, yes and no. Let's see what happens from each perspective.

It's tempting to think about how fast second hands on clocks might be moving, but Spock and Sarin can't see each other's clocks. What they could see, though, would be a signal timed to flash every second. Each could measure the time between signal blips emitted by the other ship. Since the time between blips is the duration between two events, that's the situation when the Lorentz equation applies to tell us about time dilation. All we have to do is decide which observer is stationary and which is moving.

The problem is that they are *both* moving. However, as we shall see, the choice of which is stationary also depends on *what they are observing relative to where they are.*

Logically–Spock is always logical, after all–Spock would be measuring the time between blips he sees coming from Sarin's ship. Spock knows that Sarin's ship sends the blips in one-second intervals because, well, Spock knows everything. But when he measures the signal from Sarin's ship and compares it to his local clock, he sees the blips seem to be more than one second apart. That says that time is moving "slower" for Spock than it is for Sarin. That makes sense: he's in motion relative to Sarin and that's what the Lorentz equation says should happen.

Now reverse things. Sarin is a Vulcan, too, so he knows and does stuff the same as Spock. So, he looks at signals from the Enterprise, and he sees **those** signals coming in more than one second intervals. That says time is moving slower for Sarin than it is for Spock!

What? That looks like a paradox, but it's not. It's because the facile understanding of time dilation we started with is still incomplete.

What's Missing.

Let's go back and look more closely at what Spock is doing. He's observing the blips emitted by Sarin's ship. Sarin's ship *times* those blips to be one second apart. That means there are two measurements going on. One measurement is on Sarin's ship, using the timer on his ship. The other measurement is the one Spock takes, using the clock on his ship.

The events at Sarin's end–the blips–*and* the timer *creating* the blips aren't moving with respect to each other–they are **stationary** with respect to each other. At Sarin's end, the "observer" isn't Sarin; it's the *timer* and the *emitter* which are in the same frame that's emitting the blips. That frame is *stationary* with respect to the events and provides a *rest state* for measuring the interval between the blips. That makes the frame at Sarin's end the *stationary frame*, and the frame that Spock is in the *moving* frame.

When Spock takes his measurements, *he's* the one that's moving with respect to the timer *and* the blips that are both in the stationary frame. Thus, he's the moving observer, and he sees the blips take longer than one second, just as the equation predicts.

Sarin, on the other hand, isn't observing *his* blips. He's observing the ones on the Enterprise. His observations exactly mirror those of Spock, except that two of them are observing *different things*. Sarin also sees blips that take longer than one second, but he's looking at the blips emitted by the Enterprise, not at the blips right next to him.

There's no paradox because they are observing different things.

So, what's missing in the above supposed paradox is the *location of the observed event*. What's implicit in the Lorentz equation is a *stationary frame*–or *rest frame*–that includes *both the event and a time keeper*. It's the crucial feature that's omitted from the facile understanding of time dilation that we started with.

The comparison of durations that the basic Lorentz equation gives is always between an observer who is in motion against this stationary or rest frame, the one that includes *both the events being observed* and an *observer measuring time*. So, there's no paradox after all.

It can get more complicated, of course.

Complications.

At this point, it kind of looks like the time dilation thing is just a matter of perspective. Yet, we know it's real. It's not just the GPS satellites–there's even deeper evidence from things like muon decay. But, still, how do we decide which time frame is stationary and which isn't?

The issue came up in a story I recently read. The characters were on a ship moving at close to lightspeed away from Earth, and one of the characters said, "Earth moves more slowly than we do, so time there goes by faster." Uh, well, that's not quite right, is it? The equation gives us the time dilation between a moving object and the *rest frame*, which may–or may not–be itself in motion. The rest frame contains both the events being observed and the measurement of duration between those events; the events and the observer taking the events are *stationary* with respect to each other. If they are moving, they are moving *together*.

Another complication arises when two observers moving at different rates come back together. Those observers will have traced out different paths through spacetime, via different patterns of acceleration and deceleration. How those paths and the relative "ages" of the observers work out will depend on this more complex, four-demensional problem and is beyond the scope of this short essay.

However, in some cases it's possible to figure things out. In 1956, Robert Heinlein wrote *Time for the Stars*, a novel about two identical twins. One stayed on Earth and the other went on a spaceship to the stars. The star-faring ship traveled at speeds nearing the speed of light. The plot thickens when the star-faring twin returns to Earth, having aged a couple of years, and finds his twin has aged over sixty years. Is that really how it would work?

Consider the space-faring twin and ignore ticking clocks. That twin has aged two years between the time he left Earth and returned. His *body* is his clock.

But the stay-at-home twin? He's been on Earth the whole time. Moreover, the events in question, the departure and return of the ship, happen on Earth. So the stationary frame has to be the stay-at-home twin's frame. Thus, the Lorentz equation asserts that one second of time, as measured by the space-faring twin, is *longer* than one second of time as measured by the stay-at-home twin.

This means, for example, each year the star-faring twin spends at 90% of the speed of light corresponds to 2.3 years for the stay-at-home twin. Each *month* at 99.9% of the speed of light corresponds to about 1.9 *years* for the stay-at-home twin. If the star-faring twin traveled one month at 99.999% of the speed of light, it would correspond to over eighteen years for the stay-at-home twin.

So, yes, the twins having different physical ages fits with the Lorentz equation. The faster the space-faring twin goes, the more time slows down for him.

Notice that no matter how much the space-faring twin's ship accelerates, it can't get to the speed of light. Every time it accelerates, time slows in the ship's frame. It can never catch up with a light beam.

A Thought Experiment.

We claimed above that we could deduce a special case of the Lorentz equation using a thought experiment. So, here goes.

Let's start by imagining an astronaut in orbit above the Earth. The astronaut has a device that consists of an *emitter* that sends out a beam of a light at a mirror and a *receiver* that senses when the reflected beam returns. She also has an extremely sensitive *timer* that measures the duration between sending the beam and receiving the reflection. The astronaut, the emitter, the receiver, and the timer are all moving together in orbit so they are *stationary* with respect to each other.

According the Lorentz equation, any other observer looking at this closed system will measure a longer duration between emission and receipt than the Astronaut. Let's see if we can figure out why.

Our astronaut will observe how long it takes for a photon to go from the emitter to the mirror and back again to the receiver. If the distance to the mirror is D, then the photon travels twice that distance, $2D$, to get there and back. But distance equals rate times time, so we know how long it takes for the photon to do that! The time t that it takes can be deduced from

$$2D = rt = ct$$

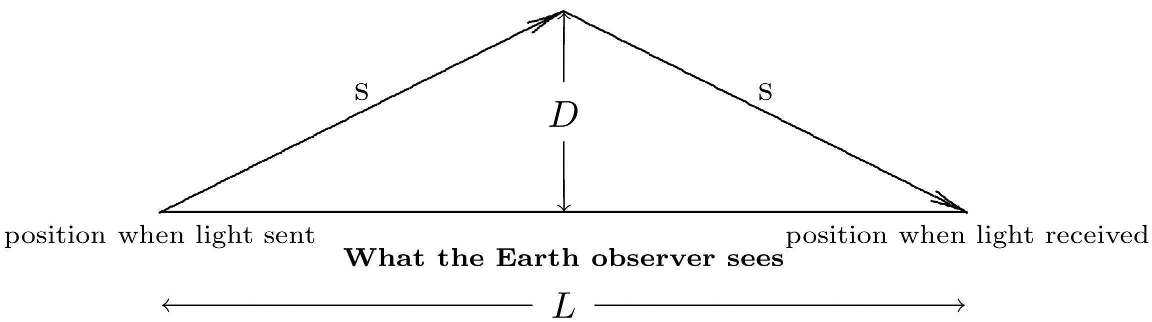
where c is the speed of light. From this, we can deduce that the time the Astronaut measures, call it $\Delta\tau$, must satisfy the equation

$$2D = c\Delta\tau \tag{1}$$

Now imagine an observer on Earth looking at the astronaut and her emitter and receiver. The earth observer is outside of the stationary frame. In fact, the relative motion between the earth observer and astronaut is due to the astronaut's orbital velocity v. So, what happens when the *Earth observer* looks at the emitter and reciever?

For the astronaut, the photon follows a straight-line path of length D to the mirror and back because, from the astronaut's perspective, the emitter and receiver haven't moved.

But things are different for the Earth-bound observer. For *him*, the emitter moves between the time of emission and the time of return. Thus, the photon does *not* follow a straight up-and-down path to the mirror, but a longer path:



What the Earth observer sees

For the Earth-bound observer, the photon follows the *longer path*, going s distance from the emiter to the mirror and then returning the same distance to the reeiver at its new position.

Since the photon must follow a longer path, it will take more time, so the Earth-bound observer will have a longer measure of elapsed time, δt that the astronaut. This is consistent with the prediction of the Lorentz equation. But we can actually *deduce* the precise Lorentz equation in this case using Euclidian geometry!

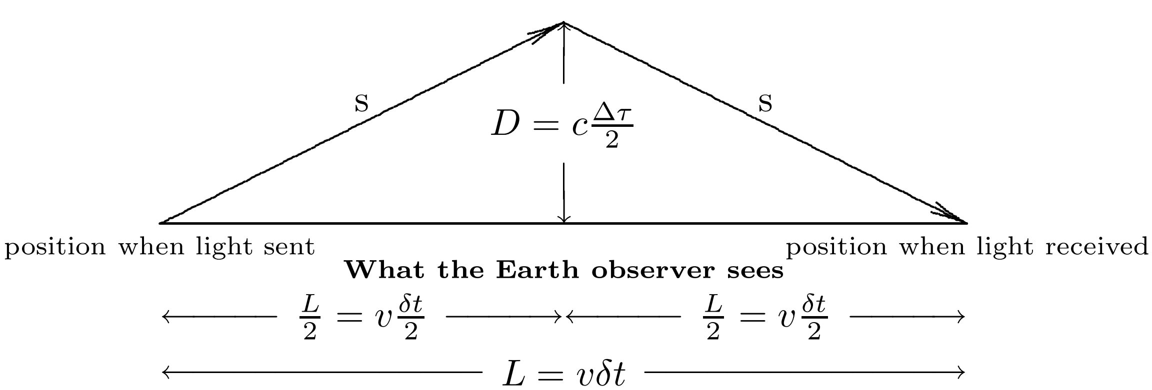
The Earth-bound observer sees δt as the durataion between emission and reciept. How far does the reciever *move* during this time? Well, if the orbital velocity of the astronaut's frame is v, then our old friend distance equals rate times time gives us

$$L = v\delta t.$$

Because of equation (1) above, we know that

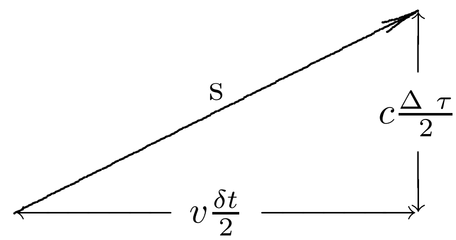
$$D = c\frac{\Delta\tau}{2}.$$

Filling this information into our picture above, we get



What the Earth observer sees

Now look at the right triangle on the left side of the diagram.



We know from the Pythagoran Thereom that

$$\left(v\frac{\delta t}{2}\right)^2 + \left(c\frac{\Delta\tau}{2}\right)^2 = s^2. \tag{2}$$

The light beam takes δt to make its trip along the two arrows, so the trip from the start position to the mirror takes half that time. Using distance equals rate times time again, we get

$$s = c\frac{\delta t}{2}$$

Substitute this into equation (2) above:

$$\left(v\frac{\delta t}{2}\right)^2 + \left(c\frac{\Delta\tau}{2}\right)^2 = \left(c\frac{\delta t}{2}\right)^2.$$

Now gather terms:

$$\begin{aligned}\left(c\frac{\Delta\tau}{2}\right)^2 &= \left(c\frac{\delta t}{2}\right)^2 - \left(v\frac{\delta t}{2}\right)^2 \\ &= \left(\frac{\delta t}{2}\right)^2 (c^2 - v^2)\end{aligned}$$

Now solve for δt:

$$
\begin{aligned}
\delta t &= 2\sqrt{\frac{\left(c\frac{\Delta\tau}{2}\right)^2}{(c^2 - v^2)}} \\
&\qquad \text{factor out } (\Delta\tau/2)^2 \\
&= 2\frac{\Delta\tau}{2}\sqrt{\frac{c^2}{c^2 - v^2)}} \\
&\qquad \text{divide numerator and denominator by } c^2 \\
&= \Delta\tau\sqrt{\frac{1}{1 - \frac{v^2}{c^2}}} \\
&= \gamma\Delta\tau
\end{aligned}
$$

which is exactly the Lorentz equation.

The only premise we made in the above is that the speed of light is the same in all frames. We can't know, but it seems likely Einstein must have done a similar thought experiment that led him to the theory of special relativity. It's pretty remarkable that a special case of a deep result like the Lorentz equation follows from Euclidian geomtry and the premise that the speed of light is the same in all frames.